Projective and Non-Abelian SET

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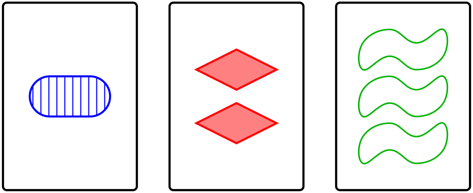
University of Bristol

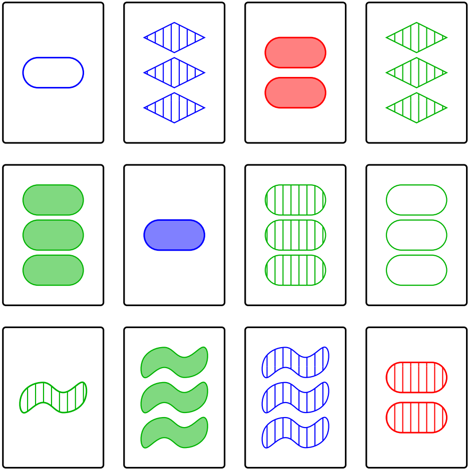
Illustrating Number Theory and Algebra
ICERM
October 24, 2019

Joint work with Jonah Ostroff and Lucas Van Meter*

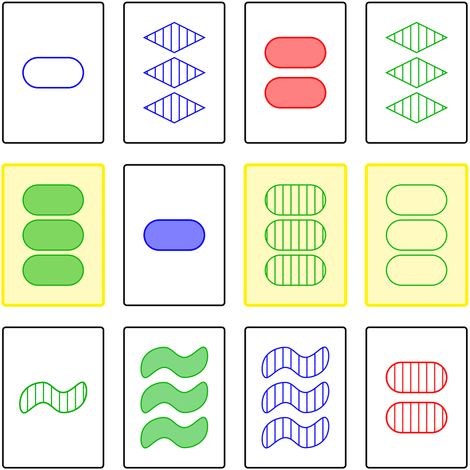
In 1974, Marsha Falco invented the card game SET:



Color	Number	Shape	Shade
red	one	diamond	solid
blue	two	squiggle	striped
green	three	oval	open



A set is a collection of three cards for which in each of the four qualities, the cards are all the same or all different.



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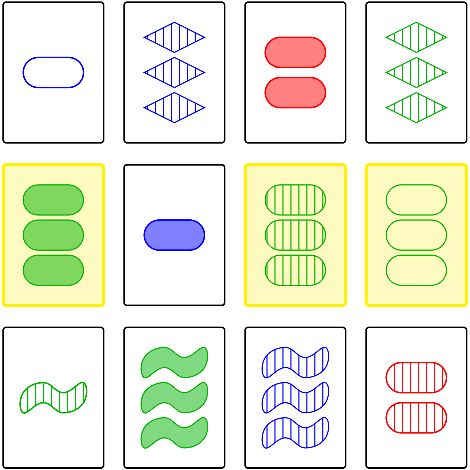

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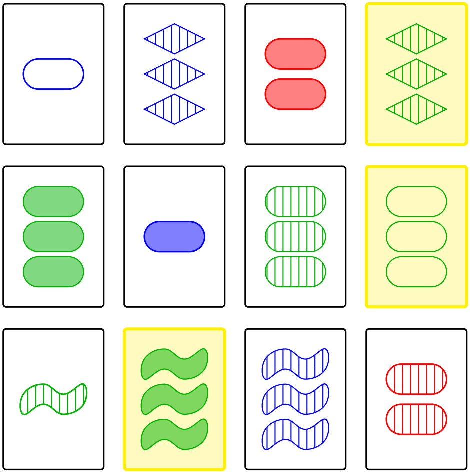
Try to find another set!



A set is a collection of three cards for which in each of the four qualities, the cards are all the same or all different.

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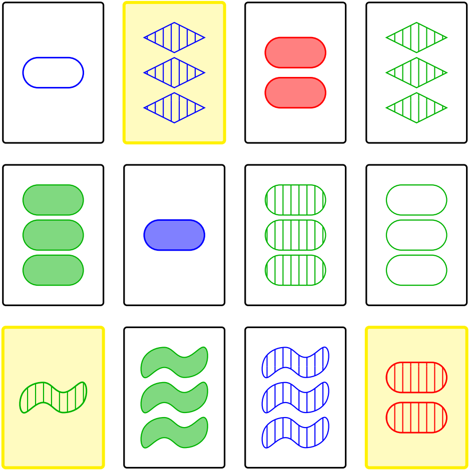

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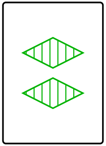
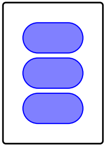
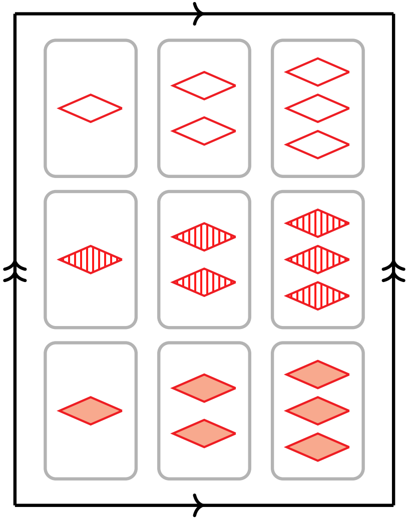
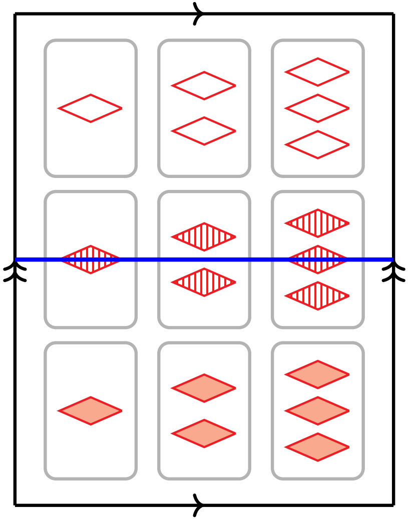


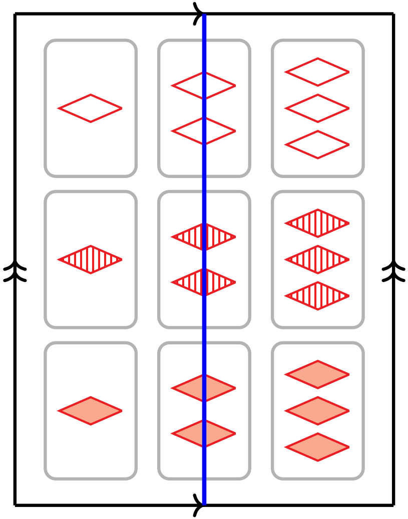
$! \quad (2; 1; 0; 1)\ 2\ F_3^4$

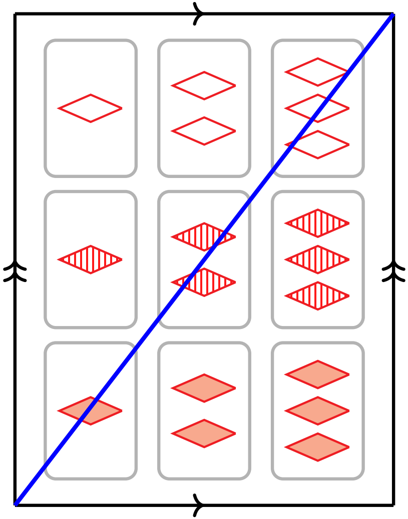


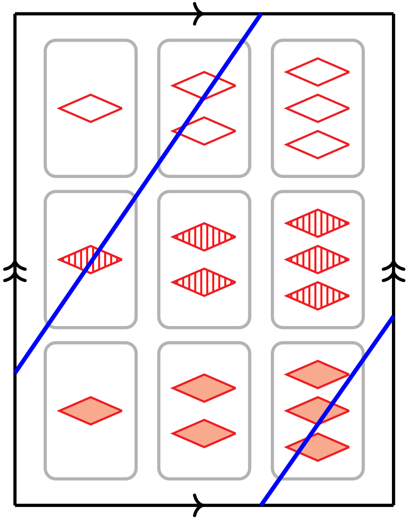
$! \quad (1; 2; 2; 2)\ 2\ F_3^4$











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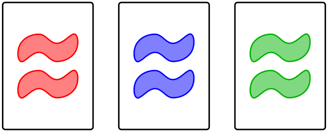

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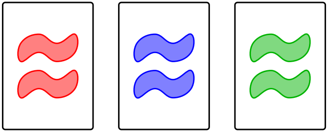

$$m$$

$$(0; 1; 1; 2) \; + \; (1; 1; 1; 2) \; + \; (2; 1; 1; 2) \; = \; 0 \; 2 \; \mathrm{F}_3^4$$

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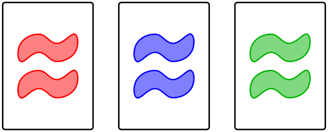

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$$\underbrace{(0,1,1,2)}_{a} + \underbrace{(1,1,1,2)}_{b} + \underbrace{(2,1,1,2)}_{c} = 0 \in \mathbb{F}_3^4$$

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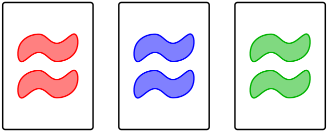

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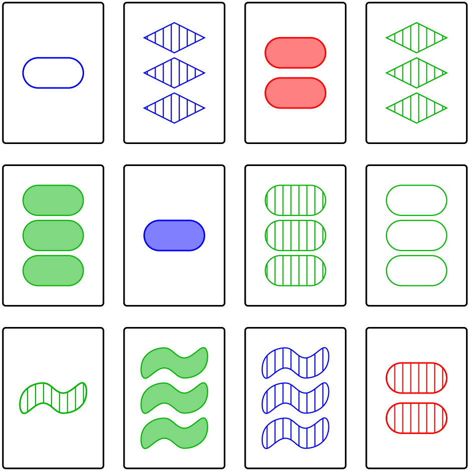
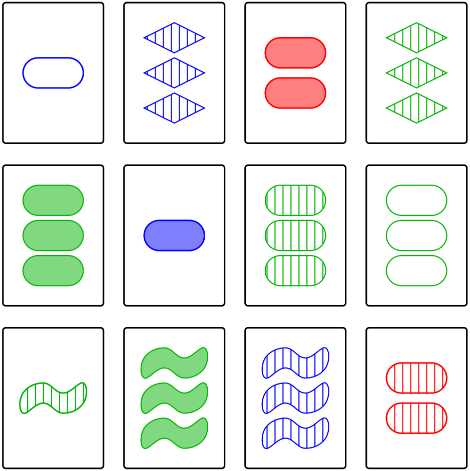
$a, b, c \in \mathbb{F}_3^4$ are collinear



New definition of a set:
a collection of four cards that sum to zero in $\mathbb{F}_3^4$.

Try to find a set!



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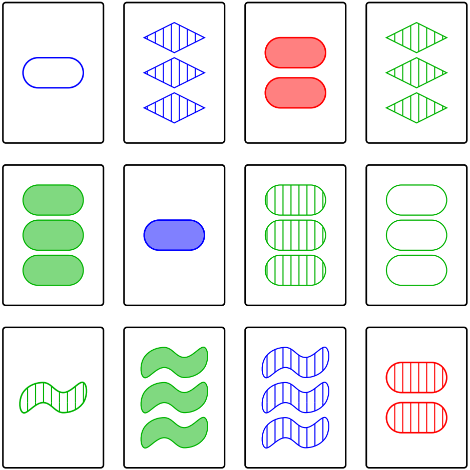
Wait! We need more info...



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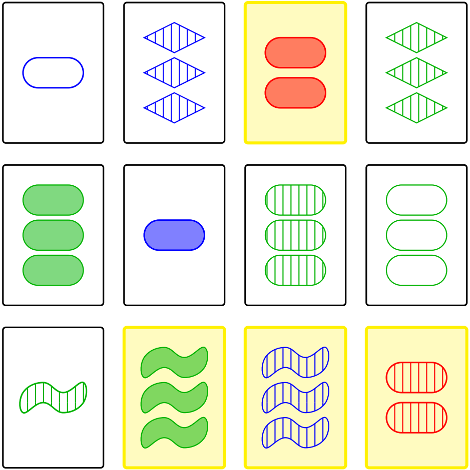

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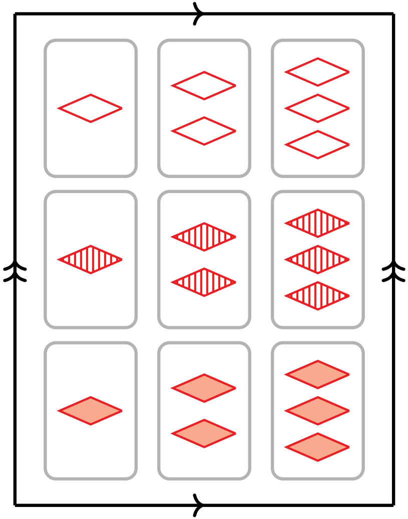
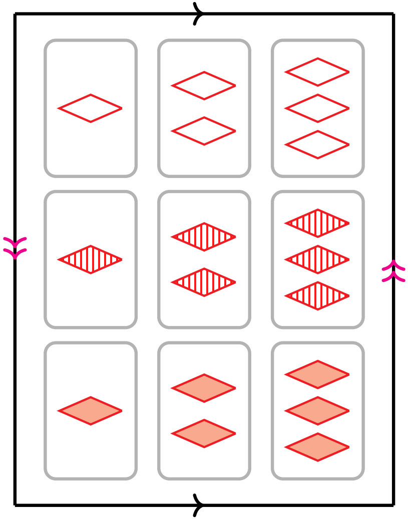
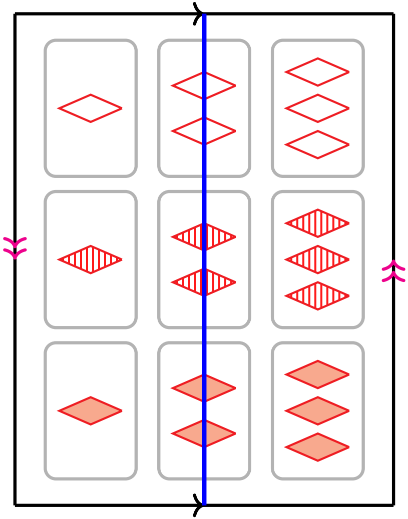
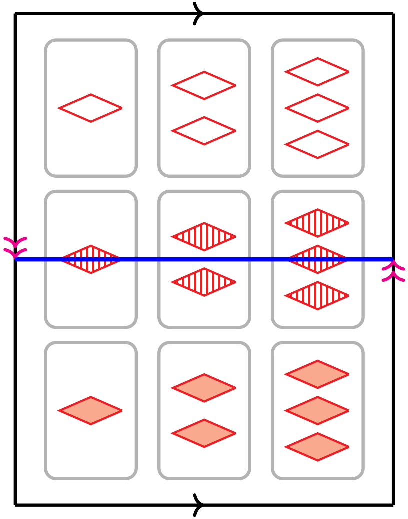
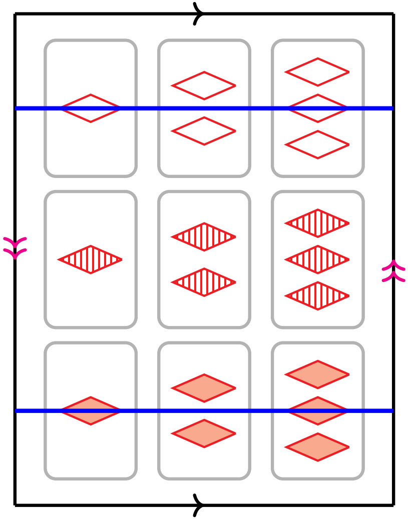
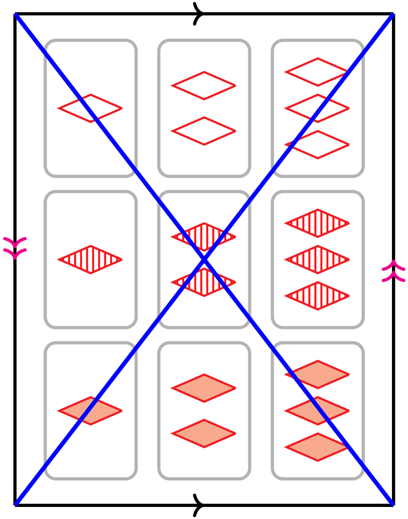


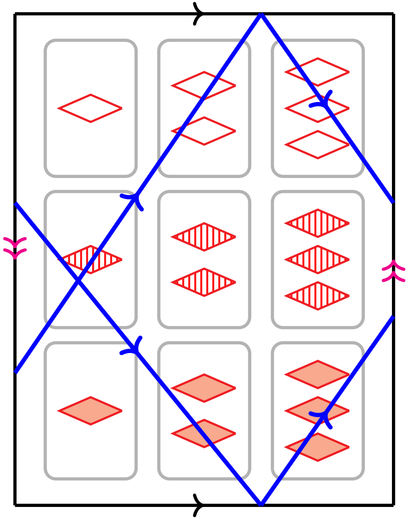


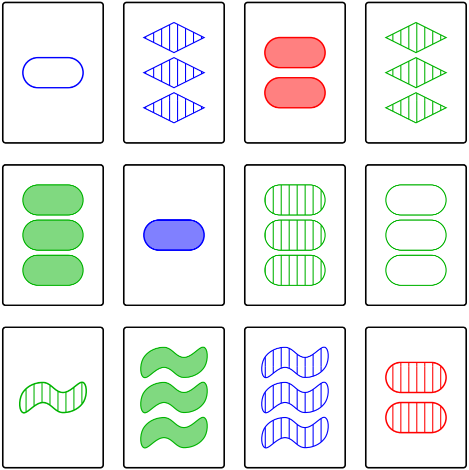




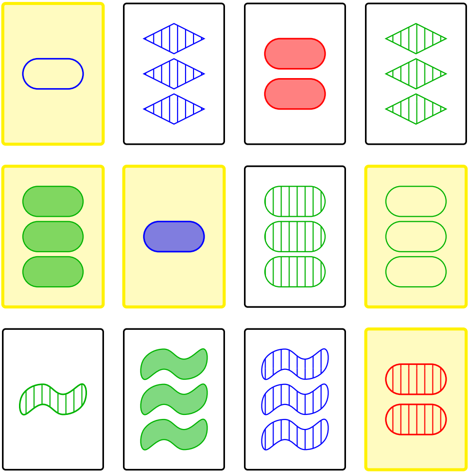








New definition of a set:
a collection of three cards that are collinear in the Möbius identification of F_3^4.



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a collection of elements that "sum" to zero

algebraic

a collection of elements that are collinear

geometric

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- Our perspective focuses on practical play.

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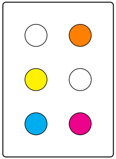
algebraic		geometric

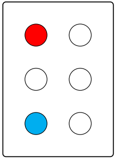
- Our perspective focuses on practical play.
- We seek interesting visual conditions for a SET.

Variation I: Projective space $\mathbf{P}^n(\mathbf{F}_2)$

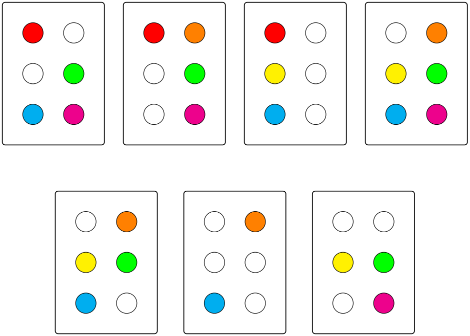
Let's take $P^5(F_2)$ as our underlying structure:



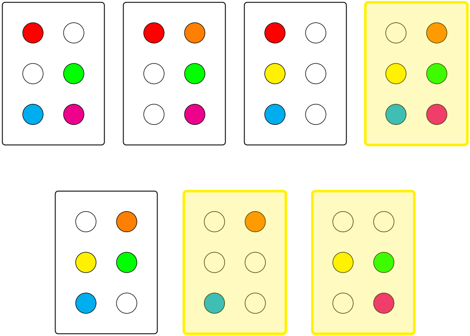
$! \quad (0; 1; 1; 0; 1; 1)\ 2\ P^5(F_2)$



$! \quad (1; 0; 0; 0; 1; 0)\ 2\ P^5(F_2)$



A projective set is a collection of three cards for which there's an even number of dots for each color.



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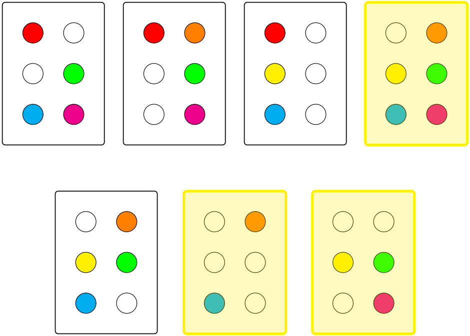
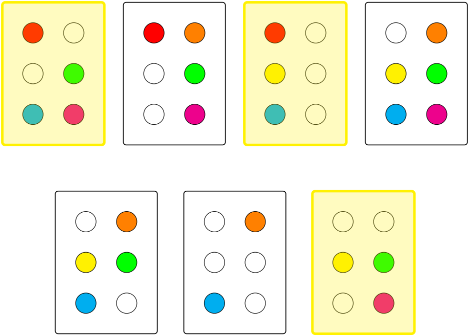


A projective set is a collection of three cards for which there's an even number of dots for each color.

Try to find another set!



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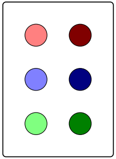
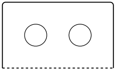
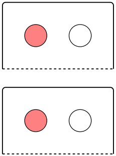


$\to \quad ((1,1),(1,1),(1,1)) \in \mathbb{P}^5(\mathbb{F}_2)$

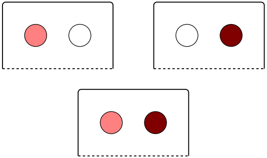
In $\mathbb{F}_2^2$, there are three ways to have an even number dots of each color:



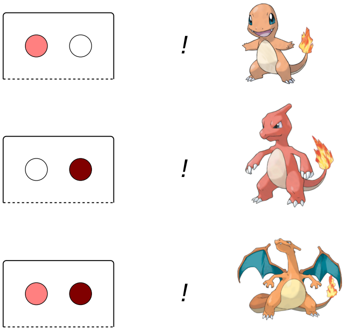
zero element



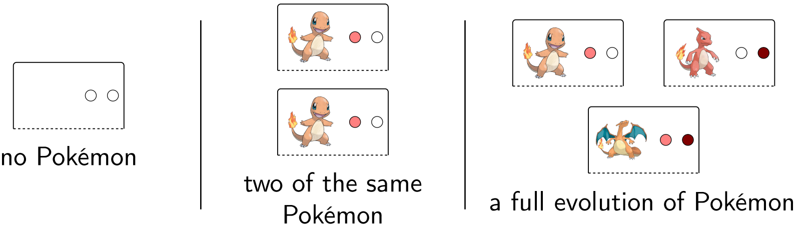
two cards with same coloring



one of each non-zero card



So, an even number dots can be appear as:



A Pokémon projective set is a collection of three cards for which the Pokémon can be partitioned into identical pairs or full evolutions.

Try to find a set!

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$\mathbb{F}_3^4$ vs. $\mathbb{P}^5(\mathbb{F}_2)$

Deal size	3-set in $\mathbb{F}_3^4$	3-set in $\mathbb{P}^5(\mathbb{F}_2)$
3	0.01	0.02
4	0.05	0.06
5	0.12	0.16
6	0.23	0.30
7	0.39	0.48
8	0.54	0.65
9	0.71	0.80
10	0.83	0.91
11	0.92	0.96
12	0.96	0.99
13	0.99	0.99
⋮	⋮	⋮

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Deal size	3-set in $\mathbb{F}_3^4$	3-set in $\mathbb{P}^5(\mathbb{F}_2)$
3	0.01	0.02
4	0.05	0.06
5	0.12	0.16
6	0.23	0.30
7	0.39	0.48
8	0.54	0.65
9	0.71	0.80
10	0.83	0.91
11	0.92	0.96
12	0.96	0.99
13	0.99	0.99
⋮	⋮	⋮

Let's redefine a Pokémon projective set to be a collection of three or more cards for which the Pokémon can be partitioned into identical pairs or full evolutions.

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Try to find a set of size 4.

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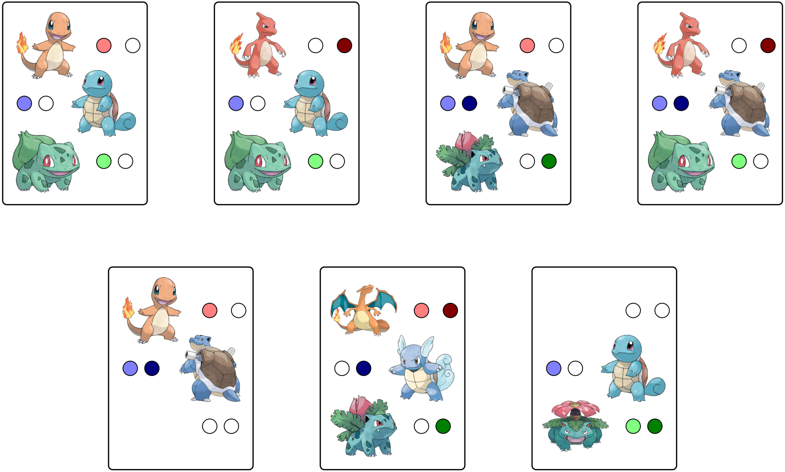
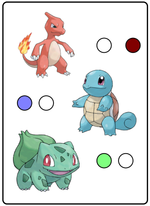
Try to find a set of size 4.

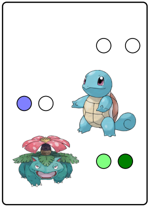
$P^5(F_2)$

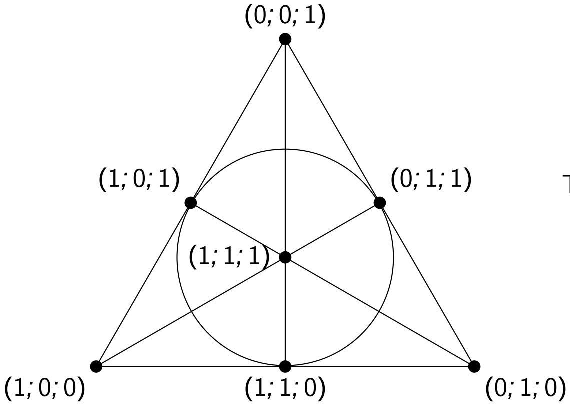
Deal size	3-set	4-set	5-set	6-set	7-set
3	0.02	–	–	–	–
4	0.06	0.02	–	–	–
5	0.16	0.08	0.02	–	–
6	0.30	0.23	0.09	0.02	–
7	0.48	0.50	0.31	0.11	0.02
8	0.65	0.76	0.61	0.39	0.12
9	0.80	0.98	0.87	0.81	0.49
10	0.91	0.99	0.97	0.98	0.92
11	0.96	0.99	0.99	0.99	0.98
12	0.99	0.99	0.99	0.99	0.99
13	0.99	0.99	0.99	0.99	0.99
⋮	⋮	⋮	⋮	⋮	⋮



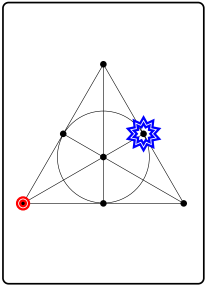





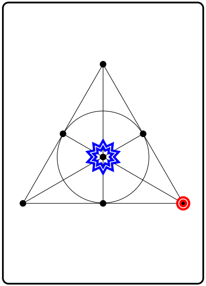





This is the Fanoplane
$P^2(F_2)$.



! $((1;0;0);(0;1;1))$



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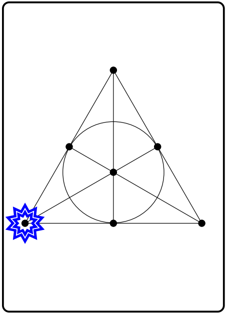
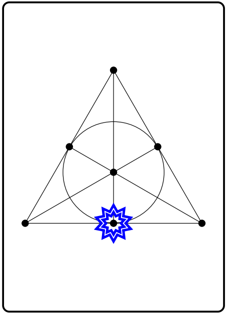
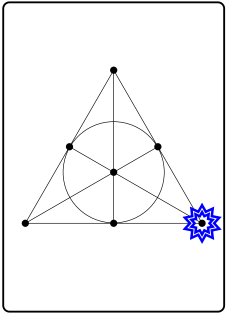
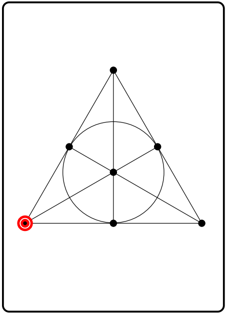
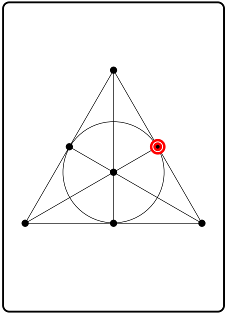
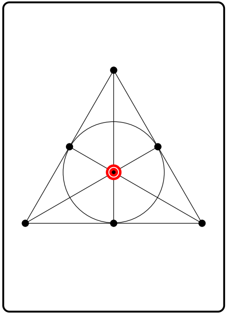
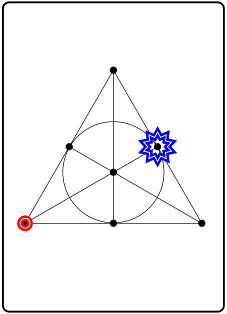
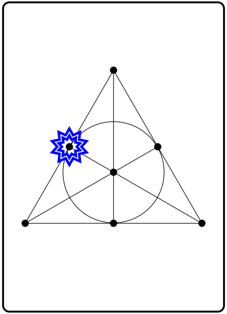
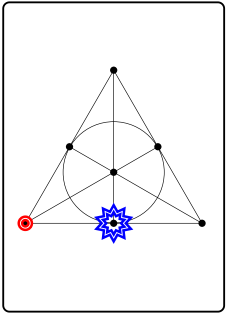


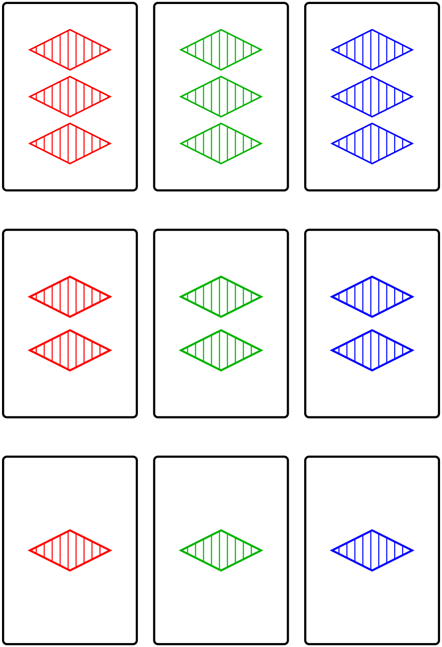


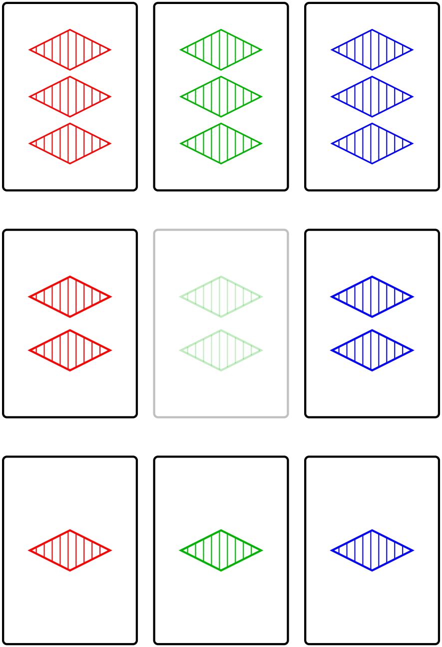


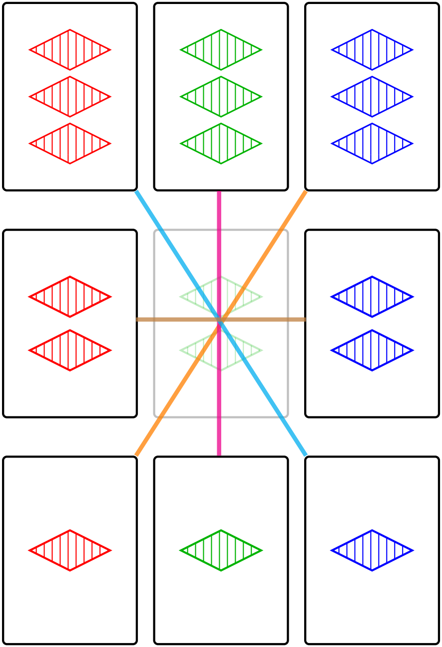




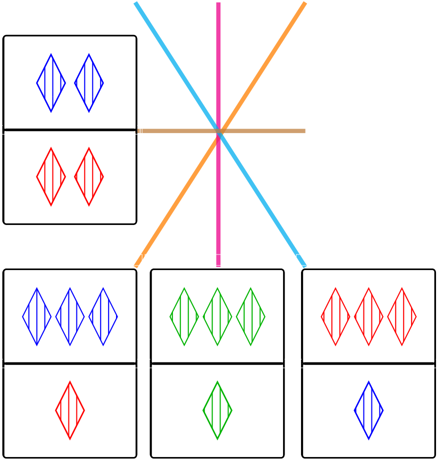




Variation II: Projectivized Normal SET $\mathbf{P}^3(\mathbf{F}_3)$









Red

Blue

!

!

Squiggle

Oval

One

Three

!

!

Open

Solid

Blue

Red

!

!

Oval

Squiggle

Three

One

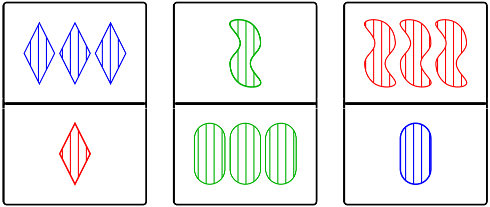
!

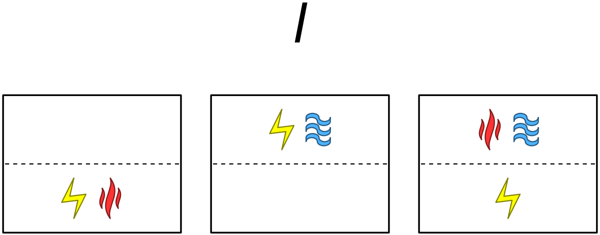
!

Solid

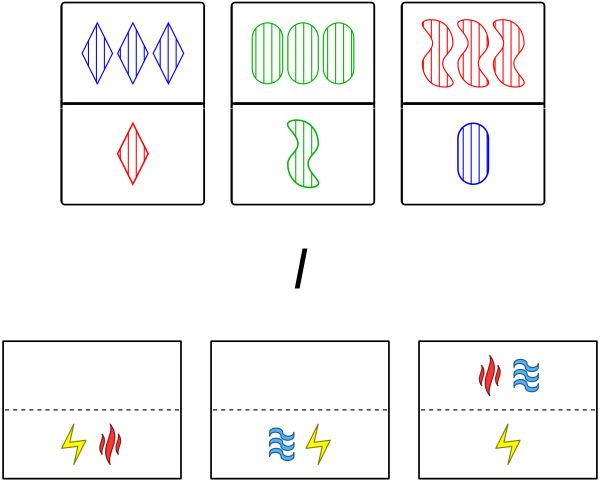
Open

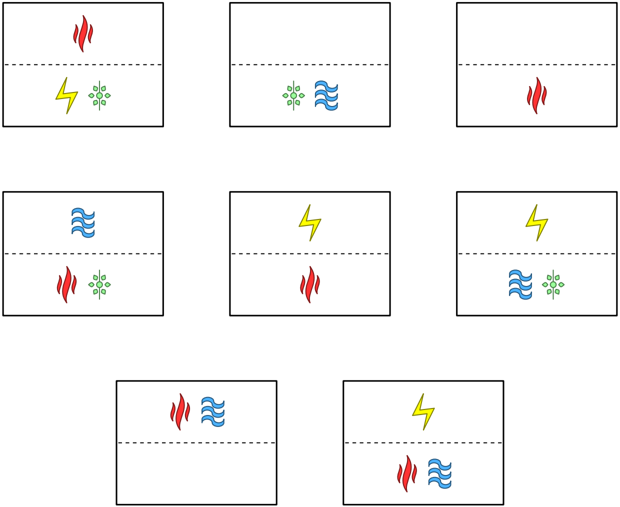
Three cards are collinear exactly when you can rotate one to obtain (normal) SETs on top and bottom. Equivalently, when each symbol appears on either all the same side or on all different sides.

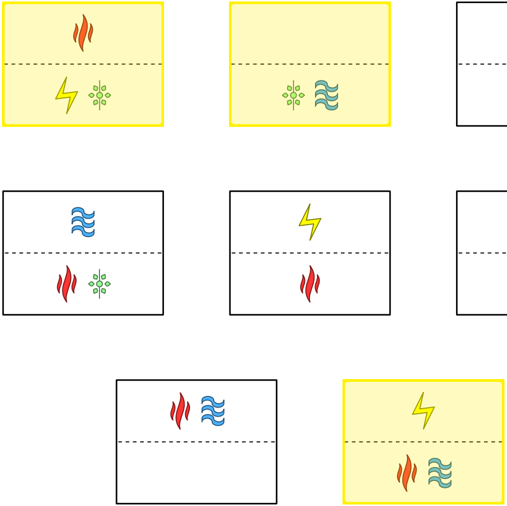


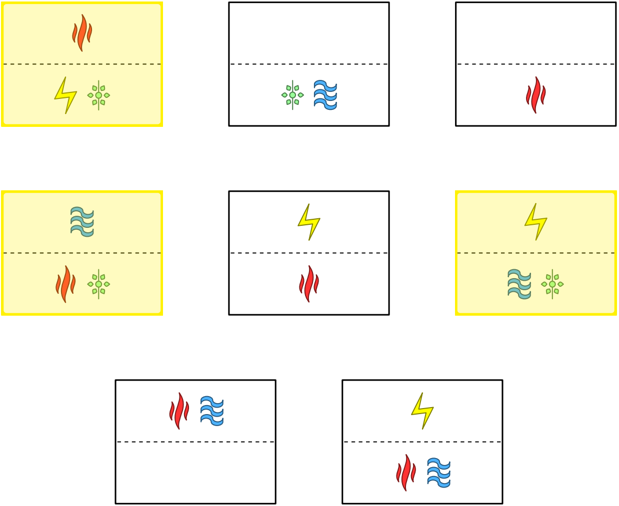


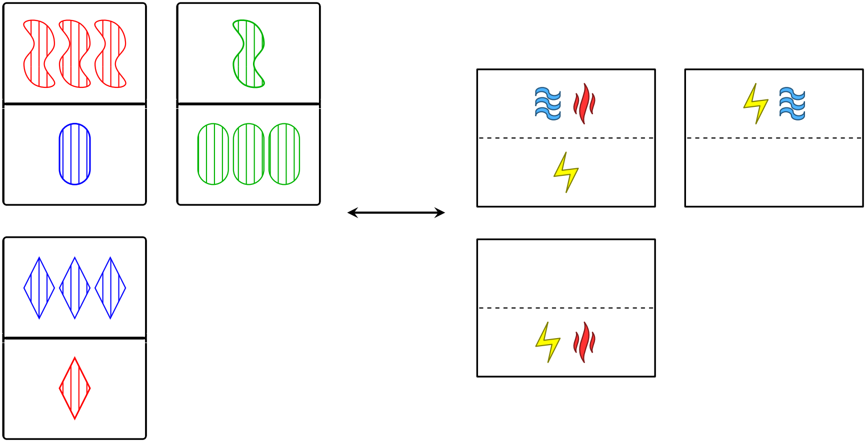
Three cards are collinear exactly when you can rotate one to obtain (normal) SETs on top and bottom. Equivalently, when each symbol appears on either all the same side or on all different sides.



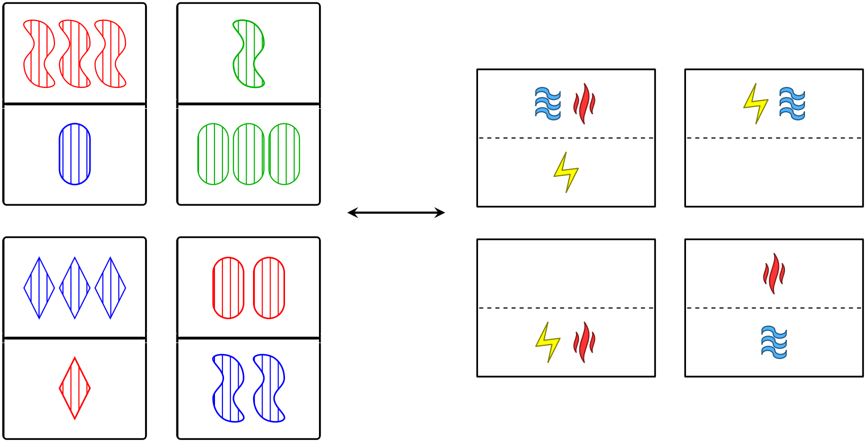








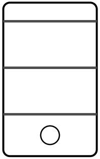
Can you figure out the missing point on this projective line?



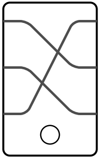
Can you figure out the missing point on this projective line?

Variation III: Non-abelian groups

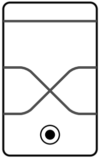
Let's start by taking the symmetric group S_3 as our underlying structure:



! id *2* S_3

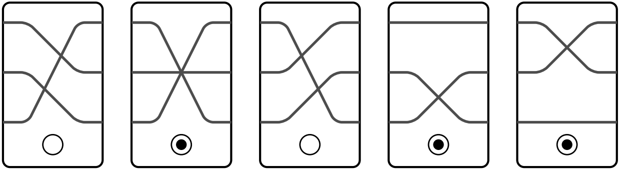


! (123) *2* S_3

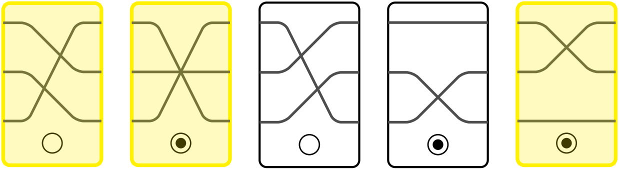


! (23) *2* S_3

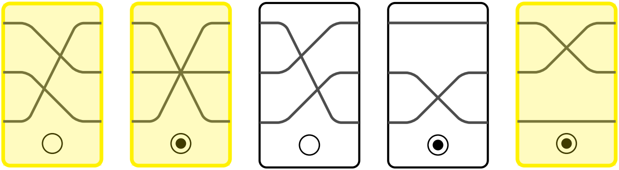
We deal the 5 non-trivial cards (which determines an ordering):



We deal the 5 non-trivial cards (which determines an ordering):



We deal the 5 non-trivial cards (which determines an ordering):



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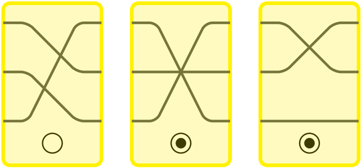
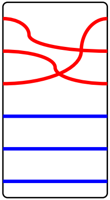
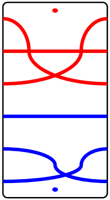


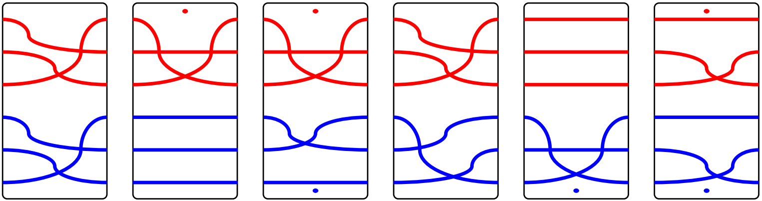
Now, let's try S_3 S_3 as our underlying structure:

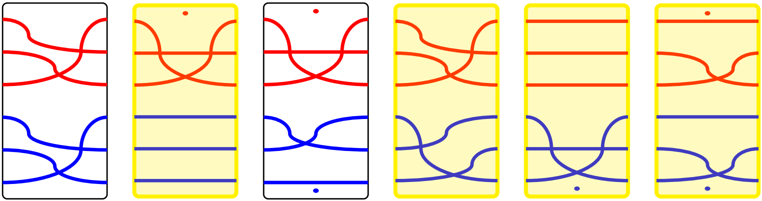


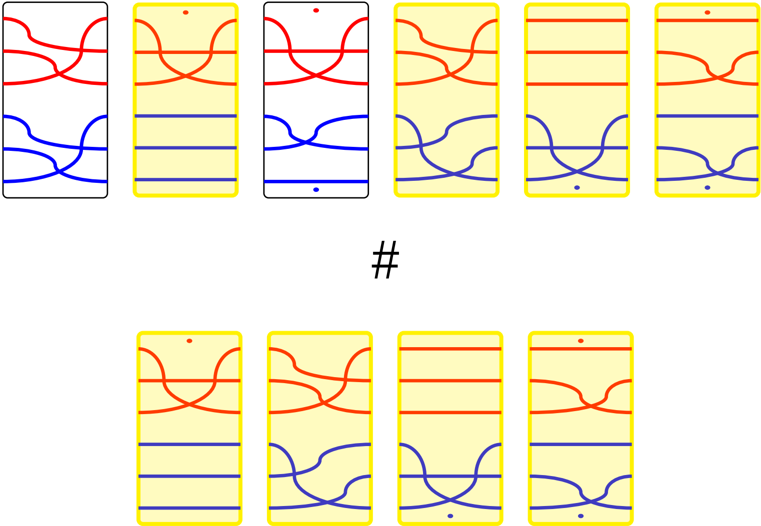
! $((123); \mathrm{id})$ 2 S_3 S_3



! $((13); (23))$ 2 S_3 S_3



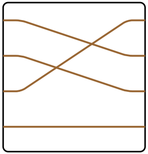




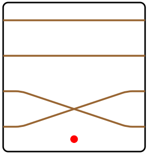
S_3 S_3

Deal size	3-set	4-set	5-set	6-set	7-set	8-set	9-set
3	0.03	–	–	–	–	–	–
4	0.11	0.03	–	–	–	–	-
5	0.27	0.14	0.03	–	–	–	–
6	0.49	0.38	0.15	0.03	–	–	–
7	0.69	0.67	0.46	0.19	0.03	–	–
8	0.87	0.91	0.83	0.58	0.20	0.03	–
9	0.95	0.99	0.97	0.92	0.67	0.23	0.03
10	0.99	0.99	0.99	0.99	0.96	0.76	0.25
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

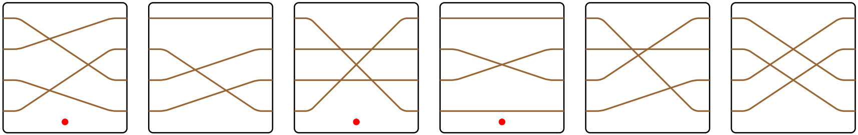
Next, we'll try S_4 as our underlying structure:

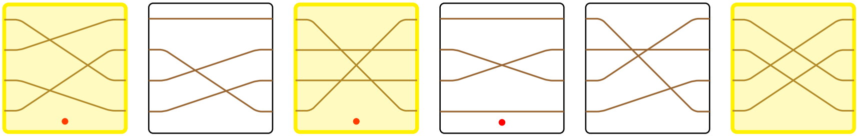


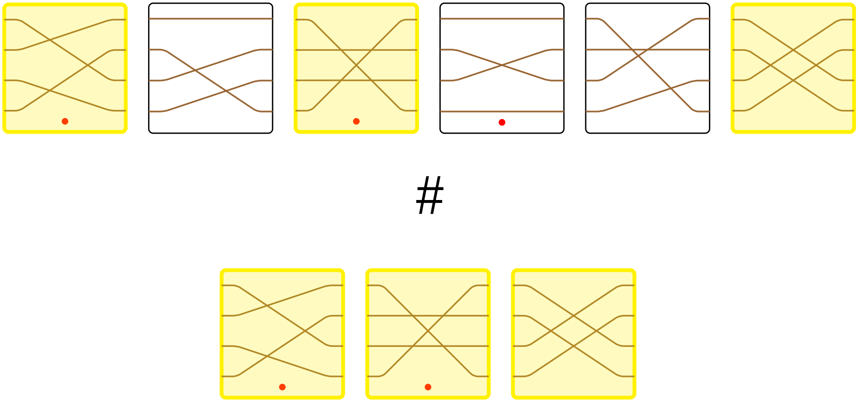
! (1 2 3) *2* S_4



! (3 4) *2* S_4



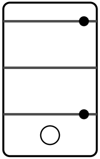




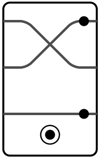
S_4

Deal size	3-set	4-set	5-set	6-set	7-set	8-set	9-set
3	0.05	–	–	–	–	–	–
4	0.17	0.04	–	–	–	–	–
5	0.40	0.21	0.04	–	–	–	–
6	0.66	0.51	0.23	0.04	–	–	–
7	0.86	0.84	0.62	0.27	0.04	–	–
8	0.97	0.98	0.92	0.74	0.30	0.03	–
9	0.99	0.99	0.99	0.98	0.82	0.32	0.04
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

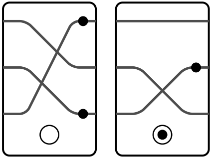
Finally, let's consider the wreath product $(\mathbb{Z}=2\mathbb{Z}) \wr S_3$:



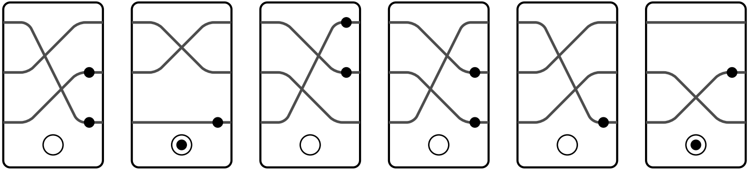
$! \quad ((1; 0; 1); \mathrm{id}) \; 2 \; (\mathbb{Z}=2\mathbb{Z}) \wr S_3$

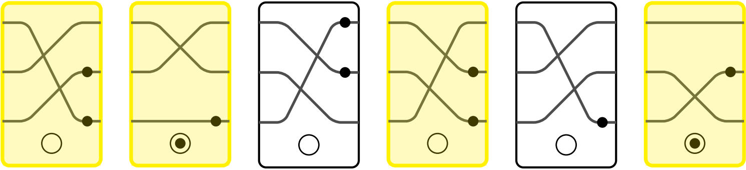


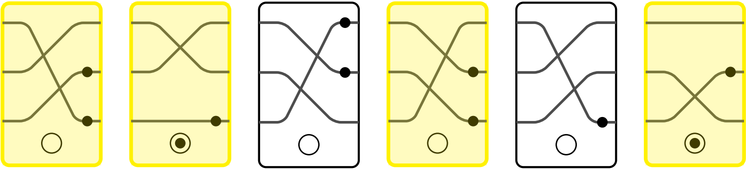
$! \quad ((1; 0; 1); (12)) \; 2 \; (\mathbb{Z}=2\mathbb{Z}) \wr S_3$



$! \quad ((1; 0; 0); (13)) \; 2 \; (\mathbb{Z}=2\mathbb{Z}) \wr S_3$







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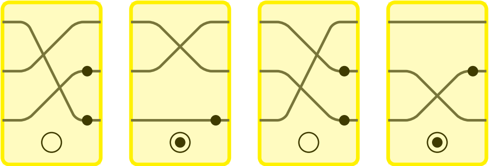

All SET decks can be found on my webpage:
https://people.maths.bris.ac.uk/ zx18363/